$\aleph_0$-CATEGORICAL GROUPS AND THEIR COMPLETIONS

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ABSTRACT [1] [2] [3] We embed a countably categorical group G into a locally compact group $\overline{G}$ with a non-trivial topology and study how topological properties of $\overline{G}$ are connected with the structure of definable subgroups of G .

1. Introduction

Using the paper of Belyaev [5] we show that any infinite countably categorical group G has an action on some countable set X which induces an embedding of G into $Sym(X)$ such that the closure of G in $Sym(X)$ is a locally compact subgroup $\overline{G}$ with non-trivial topology. We call the group $\overline{G}$ the *completion* of G (with respect to the action). We will see that when G is residually finite, the construction of the action (G, X) has the property that the completion $\overline{G}$ is the profinite completion of G.

This provides new tools in the subject. In particular we show that there are strong connections between the lattice of 0-definable subgroups of G and topological properties of $\overline{G}$. In particular the cofinality of $\overline{G}$ plays an essential role in the relationship between G and $\overline{G}$ (see Section 3).

We remind the reader that $\overline{G}$ is of *countable cofinality* if $\overline{G}$ can be presented as the union of an increasing ω-chain of proper subgroups. This property frequently arises

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in ivestigations of automorphsm (homeomorphism) groups of countable structures (Polish spaces). We recommend the reader the paper [20] (and the recent paper of Ch.Rosendal [18]) for a small survey of the subject.

In Section 4 we study the question if $\overline{G}$ can be ω-categorical. Here we concentrate on the case when G is soluble. Proposition 4.1 partially explains to us why this case is so important. Using the ideas of [3] we show that the condition that $\overline{G}$ is ω-categorical, is very restrictive. In particular it implies that when G is soluble and residually finite, the group $\overline{G}$ is nilpotent-by-finite. Coverings in the lattice of definable subgroup play an essential role both in these respects and in the case of cofinality.

In Section 5 we study measurable subsets of $\overline{G}$ with respect to a corresponding Haar measure. We characterize the case when the set of pairs of commuting elements of $\overline{G}$ has a fixed positive measure in any compact subgroup of $\overline{G}$. We also describe some aproximating function naturally arising in these respects.

It looks likely that aspects of measure and category can be exploited much further. In particular we hope that our approach can be used in some well-known problems on ω-categorical groups (BCM-conjecture and Apps-Wilson conjecture).

Our basic algebraic and model-theoretic notation is standard (the same as in [2]). For example $[x,y]$ is an abbreviation for $x^{-1}y^{-1}xy$ and $G^{(0)} = G$, $G^{(1)} = [G,G]$,...,$G^{(i+1)} = [G^{(i)}, G^{(i)}]$,... . When $\phi(\bar{x},\bar{y})$ is a formula, M is a structure and $\bar{b}$ is a tuple from M, then we denote by $\phi(M,\bar{b})$ the set $\{\bar{a} \in M : M \models \phi(\bar{a},\bar{b})\}$.

It is assumed in Section 4 that the reader is already acquainted with the most basic notions and facts of nilpotent groups and representations of finite groups (for example, commutator identities, Fitting subgroups [4] , Maschke's theorem and Schur's lemma).

We will also use some basic theory of profinite groups. Among general topological arguments (as in [22], Section 0.3), we will use profinite Sylow theory. We will permanently consider the situation when a Polish group G acts continuously on a Polish space X. Then we assume that the reader understands some obvious consequences of this situation. For example, the stabilizer of an element of the

[4] maximal normal nilpotent subgroup

space (in particular the centralizer of an element in a topological group) is closed. If G is compact then each G-orbit in X is compact and closed.

In fact our model-theoretic arguments are restricted to some basic properties of ω-categorical theories (outside the title of the paper we prefer ω-categorical to $\aleph_0$-categorical). We now give some preliminaries concerning ω-categorical groups. Let G be ω-categorical. Since all $Aut(G)$-invariant relations of G are definable, all characteristic subgroups of G are definable. It is also worth noting that for any natural k the group G has only finitely many $Aut(G)$-invariant k-ary relations. In particular G is *uniformly locally finite*: there is a function $f : \omega \to \omega$ such that every n-generated subgroup of G is of size $\leq f(n)$.

By $\langle g \rangle^G$ we denote the normal subgroup of G generated by g. We mention Lemma 3.2 from [14].

Lemma 1.1. *If G is an ω-categorical group, then there is a formula $\psi(x, y)$ such that $G \models \psi(g, h)$ if and only if $h \in \langle g \rangle^G$.*

This is a consequence of the fact that G has *finite conjugate spread* (see [2]): there exists a number m such that for any $g \in G$ any element of $\langle g \rangle^G$ is a product of $\leq m$ conjugates of g^ε, $\varepsilon \in \{-1, 1\}$.

Theorem A of [2] will play an important role in our arguments. It states that an ω-categorical group G has a finite series $1 = G_0 < G_1 < ... < G_n = G$ with each G_i characteristic in G, and with each G_i/G_{i-1} either elementary abelian, or isomorphic to some Boolean power P^B with finite simple non-abelian P and an atomless Boolean ring B, or an ω-categorical characteristically simple non-abelian p-group for some p. It is still unknown if the third case can happen (the Apps-Wilson conjecture states that it cannot happen).

Let us consider the case of Boolean powers in more detail (following [2] and [13]). Let B be a Boolean ring (that is, a commutative ring in which every element is an idempotent). Then B can be identified with the ring of compact open subsets of the Stone space $X = S(B)$ of B. We remind the reader that the space $S(B)$ consists of all maximal ideals of B, and the ring of all compact open subsets of $S(B)$ is considered with respect to symmetric difference $+$ and intersection $\cdot$.

The *Boolean power* of G by B, denoted by G^B, is defined to be the set of functions $X \to G$ which are continuous with compact support (G is disctete). When $g \in G$

and $A \in B$, then by g_A we denote the element of G^B with support A which takes each $x \in A$ to g. Then every element of G^B can be written uniquely (up to order) in the form $(g_1)_{A_1} \cdot ... \cdot (g_n)_{A_n}$ with $g_1, ..., g_n$ distinct elements of $G \setminus \{1\}$; and $A_1, ..., A_n$ pairwise disjoint elements of $B \setminus \{0\}$. A theorem of Waszkiewicz and Węgłorz claims that when G is ω-categorical and B has finitely many atoms, the group G^B is ω-categorical.

It is worth noting that there are exactly two countable atomless Boolean rings: V_0 without an identity element and V_1 with an identity element. These are ω-categorical, and any ω-categorical countable Boolean ring is isomorphic to a product of one of these with a finite field of sets.

We will concentrate on the case when G is a finite simple non-abelian group. In this case any normal subgroup of G^B is of the form G^S where S is an ideal of B (see [1], Proposition 5.1). Theorem 2.3 of [2] states that G^B does not have proper characteristic subgroups if B is finite or a countable atomless Boolean ring (i.e. $R = V_0$ or $R = V_1$).

One of the statements of Theorem B of [2] is that if a countable characteristically simple non-abelian group has a subgroup of finite index, then it is isomorphic to some P^B where P is a finite simple non-abelian group and B is a countable characteristically simple Boolean ring.

We will also use Boolean powers of rings. If R is such a ring then R^B is defined by the same definition as in the case of groups. In fact we will use *filtered Boolean powers* of rings. We remind the reader that a filtered Boolean power of R (of a group G resp.) with respect to a Boolean space $X = S(B)$, where B is a Boolean algebra, is defined by a finite sublattice L of closed subsets of X and an isomorphism τ from L to a sublattice of subrings of R (subgroups of G resp.). The corresponding filtered Boolean power is the substructure of R^B (of G^B resp.) consisting of those $f \in R^B$ such that for each $C \in L$, $f(C) \subseteq \tau(C)$. Denote this structure by $C(X, R, \tau)$ (by $C(X, G, \tau)$ resp.). Note that each closed subsets C from L naturally corresponds to some ideal I_C of B with $C = \{I \in X : I_C \subseteq I\}$. Then the structure $(B, I_C)_{C \in L}$ is called an *augmented Boolean algebra*.

The following Macintyre-Rosenstein description of ω-categorical rings (see [13]) is one of the basic results in the area.

> Let R be an ω-categorical ring with 1 with no nilpotent elements. Then R is a direct product of finitely many 0-definable rings R_i and each R_i is isomorphic to a filtered Boolean power of a finite field F_i with respect to an appropriate Boolean space X_i and a map τ_i to a lattice of subfields of F_i. The corresponding augmented Boolean algebras B_i are interpretable in R (and thus ω-categorical).

When $R = F^B$ is a Boolean power of a finite field, all ideals of R can be described in a very similar way as in the case of a Boolean power of a finite simple non-abelian group (Proposition 5.1 of [1]): any ideal of F^B is of the form F^S where S is an ideal of B. We think that this is a folklour fact. On the other hand the proof of Proposition 5.1 from [1] works for F^B with obvious changes (for example, commutators $[g_i, h] \neq 1$ should be replaced by products $g_i h \neq 0$).

2. Topological completion

A subgroup H of an infinite group G is called *inert in G* if for every $g \in G$, $H \cap gHg^{-1}$ is of finite index in H.

Proposition 2.1. *Let G be a countably infinite uniformly locally finite (for example, countably categorical) group.*
(1) Then for every finite subgroup $F < G$ there is a finite nontrivial subgroup K such that $F \leq N_G(K)$ and $K \cap F = 1$.
(2) Every finite subgroup of G is contained in an infinite resudually finite inert subgroup of G.

Proof. We use Corollary 1.7 of [5]: *A locally finite uncountable group satisfies statement (1) of the formulation.* If G contains a finite subgroup F which does not satisfy (1), then F does not satisfy (1) in any uncountable elementary extension G_1 of G. Since G is uniformly locally finite, G_1 is locally finite. Thus we have a contradiction.

Statement (2) follows fom (1) by Theorem 1.2 of [5] □

Let H be an inert residually finite subgroup of G and let

$$X = \{gK : K \text{ is commensurable with } H \text{ and } g \in G\}.$$

Since G is countable, X is countable too. By Theorem 7.1 of [5] the action of G on X by left multiplication defines an embedding of G into $Sym(X)$ such that the closure of G in $Sym(X)$ is a locally compact subgroup $\overline{G}$ and the closure of any K commensurable with H is a compact subgroup of $\overline{G}$. We call the group $\overline{G}$ the *completion* of G with respect to the G-space (G, X) (or with respect to H).

It is easy to see that when G is residually finite, the completion $\overline{G}$ (with respect to G) is the profinite completion of G. This observation can be developed as follows. Let us enumerate $X = \{\xi_1, \xi_2, ..., \xi_n, ...\}$ and present X as an increasing sequence of G-invariant subsets $X_n = G\xi_1 \cup ... \cup G\xi_n$, $n \in \omega$. Let G_n be the permutation group induced by G on X_n. Here we consider G_n as a topological group with respect to the topology of the action on X_n. The following lemma will be helpful below.

Lemma 2.2. *Assume that G is residually finite and the subgroup H of the construction coincides with G.*
(1) The group $\overline{G}$ is the inverse limit of the system of finite permutation groups G_n with respect to the system of appropriate projections (induced by restrictions).
(2) Let $N_1 < N_2 < G$ be subgroups of G with $N_1 \lhd G$. Then the inverse system $\{... \leftarrow (G_n \cap N_2)/(G_n \cap N_1) \leftarrow ...\}$ induced by $\{... \leftarrow G_n \leftarrow G_{n+1} \leftarrow ... : n \in \omega\}$ has the limit isomorphic to the quotient $\overline{N}_2/\overline{N}_1$ of the corresponding closures in $\overline{G}$.

Proof. Since each ξ_n is a coset of a subgroup of finite index, statement (1) is obvious. Now the fact that in (2) the group $\overline{N}_2$ ($\overline{N}_1$ resp.) is the inverse limit of the system $\{... \leftarrow G_n \cap N_2 \leftarrow ...\}$ (or $\{... \leftarrow G_n \cap N_1 \leftarrow ...\}$ resp.) is standard. Thus statement (2) of the lemma becomes standard too. □

We now collect some basic facts about our construction.

Proposition 2.3. *Let G be an ω-categorical group. Any finitely generated subgroup of the closure $\overline{G}$ is a homomorphic image of some finite subgroup of G. In particular, $\overline{G}$ is locally finite. If G is k-step nilpotent (soluble), then $\overline{G}$ is k-step nilpotent (soluble) too.*

Proof. We prove a slightly stronger statement. Let $\{c_1, ..., c_k, a_1, ..., a_l\}$ be a finite subset of $\overline{G}$, where $c_1, ..., c_k \in G$. We consider $\bar{c}\bar{a}$ as a tuple of permutations on X. Let $(a_{i,j})_{j\in\omega}$ be a sequence of elements of G converging to a_i in $Sym(X)$

where $1 \le i \le l$. Since G is uniformly locally finite we may assume that for all $s \neq t$ the map $a_{i,s} \to a_{i,t}$, $1 \le i \le l$, extends to a $\bar{c}$-stabilizing isomorphism between the groups $\langle \bar{c}, a_{1,s}, ..., a_{l,s} \rangle$ and $\langle \bar{c}, a_{1,t}, ..., a_{l,t} \rangle$. This obviously implies that $\langle \bar{c}, \bar{a} \rangle$ is a homorphic image of $\langle \bar{c}, a_{1,s}, ..., a_{l,s} \rangle$. The rest is obvious. $\Box$

We will also use the following lemma.

Lemma 2.4. *If in the situation above the group $\overline{G}$ is compact, then for every characteristic subgroups $K, L < G$ the commutator subgroup $[\overline{K}, \overline{L}]$ is the closure of $[K, L]$ in $\overline{G}$.*

Proof. Let $g_1, ..., g_k \in \overline{K}$, $g'_1, ..., g'_k \in \overline{L}$ and $w(g_1, ..., g_k, g'_1, ..., g'_k)$ be a word cosisting of commutators $[g_i, g'_i]$. Let g_{ij} be a sequence from K converging to g_i, $i \le k$, and g'_{ij} be a sequence from L converging to g'_i, $i \le k$. Then $w(g_{1j}, ..., g_{kj}, g'_{1j}, ..., g'_{kj})$, $j \to \infty$, converges to $w(g_1, ..., g_k, g'_1, ..., g'_k)$. Thus $w(g_1, ..., g_k, g'_1, ..., g'_k)$ belongs to the closure of $[K, L]$.

To see the converse take a sequence $g_j \in [K, L]$ converging to some $g \in \overline{G}$. Since G is ω-categorical, there is $l \in \omega$ such that each g_j can be written as a product of l commutators $[h_{1,j}, h'_{1,j}]^{\varepsilon_1} \cdot ... \cdot [h_{l,j}, h'_{l,j}]^{\varepsilon_l}$ with $h_{i,j} \in K$, $h'_{i,j} \in L$ and $\varepsilon_i \in \{-1, 1\}$. Since $\overline{G}$ is compact we may assume that the sequence of tuples $\bar{h}_j \bar{h}'_j$ is convergent to some $h_1, ..., h_l, h'_1, ..., h'_l$ from $K \cup L$. Thus $g = [h_1, h'_1]^{\varepsilon_1} \cdot ... \cdot [h_l, h'_l]^{\varepsilon_l}$. $\Box$

The following lemma shows that in the situation of Lemma 2.4 the group $\overline{G}$ also has finite conjugate spread.

Lemma 2.5. *Assume that G is ω-categorical and residually finite (thus $\overline{G}$ is profinite). There is a number m such that for any $g \in \overline{G}$ any element of $\langle g \rangle^{\overline{G}}$ is a product of $\le m$ conjugates of g^{ε}, $\varepsilon \in \{-1, 1\}$.*

Proof. Since G is ω-categorical, there is a number m such that for any $g \in G$ any element of $\langle g \rangle^G$ is a product of $\le m$ conjugates of g^{ε}, $\varepsilon \in \{-1, 1\}$. If $g \in \overline{G}$ and $g' = \prod_{i \le l} h_i^{-1} g^{\varepsilon_i} h_i$, then find sequences $g_j \in G$, $j \in \omega$, and $h_{i,j} \in G$, $i \le l$, $j \in \omega$, converging to g and h_i respectively. Then each $g'_j = \prod_{i \le l} h_{ij}^{-1} g_j^{\varepsilon_i} h_{ij}$ is a product of m conjugates of g_j^{ε}: g''_{1j}, ...,g''_{mj} from G. Since $\overline{G}$ is compact we may assume (taking a subsequence if necessary) that g''_{1j}, ...,g''_{mj} $j \in \omega$, converges to some conjugates of g^{ε}: g''_1, ...,g''_m from $\overline{G}$. Thus $g' = g''_1 \cdot ... \cdot g''_m$. $\Box$

3. Cofinality

Let G be an ω-categorical group and $\overline{G}$ be the completion of G with respect to an appropriate H as it was defined in Section 2. In this section we investigate cofinality of $\overline{G}$.

Proposition 3.1. *Under circumstances above the following statements hold.*
(1) If the group $\overline{G}$ is not compact, then $\overline{G}$ has countable cofinality in the following stronger sense: $\overline{G}$ is the union of an ω-chain of proper open subgroups ($\overline{G}$ has countable topological cofinality).
(2) If the group G is locally soluble, then $\overline{G}$ has countable cofinality.

Proof. Let H be an inert residually finite subgroup of G,

$$X = \{gK : K \text{ is commensurable with } H \text{ and } g \in G\},$$

and $\overline{G}$ be the corresponding closure.

(1) It suffices to prove that $\overline{G}$ is not compactly generated (see p.9 of [18]). Let $F \subset \overline{G}$ be compact and gK be any element of X. Since F is compact and the topology of $\overline{G}$ is defined by the topology of $Sym(X)$, the set $\{fgK : f \in F\}$ (of images of gK under F) is finite (say $f_1 gK, ..., f_l gK$). Thus $Fg \subset \bigcup f_i gK$ and F is contaned in the subgroup $K_1 = \langle f_1, ..., f_k, g, K\rangle$.

Let $K_0 = \bigcap\{K^h : h \in \langle f_1, ..., f_k, g\rangle\}$. Then K_0 is $\langle \overline{f}, g\rangle$-normalized and of finite index in K. Let $h_1, ..., h_l$ represent all nontrivial cosets of K/K_0. Since G is locally finite, the group K_0 is of finite index in $\langle \overline{f}, g, \overline{h}, K_0\rangle$. We see that K is of finite index in $K_1 = \langle f_1, ..., f_k, g, K\rangle$. Thus $K_1 \in X$ and by Theorem 7.1 of [5] the closure $\overline{K}_1$ is compact. We now see that F cannot generate $\overline{G}$.

(2) By statement (1) we may assume that G is residually finite and $\overline{G}$ is compact. In this case $\overline{G}$ is prosoluble and $[\overline{G}, \overline{G}]$ is a proper closed subgroup of $\overline{G}$ (Lemma 2.4). We claim that $\overline{G}$ is really soluble.

To see this we apply Theorem A of [2]. By this theorem G has a finite series $1 = G_0 < G_1 < ... < G_n = G$ with each G_i characteristic in G, and with each G_i/G_{i-1} either elementary abelian, or isomorphic to some Boolean power P^B with finite simple non-abelian P and atomless Boolean ring B, or an ω-categorical characteristically simple non-abelian p-group for some p. By Theorem B of [2] an infinite ω-categorical characteristically simple non-abelian p-group is not residually

finite. Thus this case cannot be realized as G_i/G_{i-1}. In the case when G_i/G_{i-1} is of the form P^B the group P is a homomorphisc image of a subgroup of G. Since G is locally finite and locally soluble, the group P cannot be simple non-abelian. As a result we see that all factors G_i/G_{i-1} are abelian and the grpoup G is soluble. By Proposition 2.3 the group $\overline{G}$ is soluble too.

Consider the chain of derived subgroups $\overline{G} > [\overline{G}, \overline{G}] > [[\overline{G}, \overline{G}], [\overline{G}, \overline{G}]] > ....$ Since $\overline{G}$ is soluble there is n such that the subgroup $\overline{G}^{(n)}$ is of finite index in $\overline{G}$, but $\overline{G}^{(n+1)}$ is of infinite index in $\overline{G}^{n}$. In this case the cofinality of $\overline{G}$ coincides with the cofinality of $\overline{G}^{(n)}$. Since an infinite abelian group has a countably infinite quotient group (Lemma 4.6 of [18]) there is a normal subgroup $L \leq \overline{G}^{(n)}$ with countable $\overline{G}^{(n)}/L$ such that $\overline{G}^{(n+1)} \leq L$. Let $g_1, ..., g_n, ...$ represent all cosets of $\overline{G}^{(n)}/L$. Since $\overline{G}^{(n)}$ is locally finite, the sequence

$$Ł \leq \langle L, g_1 \rangle \leq ... \leq \langle L, g_1, ..., g_n \rangle \leq ...$$

contains a strictly increasing chain such that its union equals $\overline{G}^{(n)}$. $\square$

Assume that G is residually finite. Consider the lattice of all characteristic subgroups of G. As we have already mentioned (in the proof above), by Theorems A and B of [2] if $N_1 < N_2$ is a *covering pair* from this lattice (i.e. there is no intermediate characteristic subgroups between N_1 and N_2), then N_2/N_1 is either elementary abelian, or isomorphic to some Boolean power P^B with finite simple non-abelian P and atomless Boolean ring B. When the former case appears, the group N_2/N_1 becomes a G/N_2-module under the conjugacy action. Note that since N_2/N_1 is abelian, this action does not depend on representatives of N_2-cosets. The same property holds for the topological $\overline{G}/\overline{N}_2$-module $\overline{N}_2/\overline{N}_1$ (by Lemma 2.2).

Under the circumstances as above we will say that N_2/N_1 has *countable* G/N_2-*cofinality* if there are a subgroup $H < G$ and an ω-chain $N_1 < U_0 < U_1 < U_2 < ...$ of subgroups of N_2 such that $G = HN_2$, $U_0 = HN_1 \cap N_2$, $\bigcup U_i = N_2$ and $(U_0/N_1) < (U_1/N_1) < (U_2/N_1) < ...$ is a proper ω-chain of G/N_2-submodules of N_2/N_1. The same definition (under the replacement $N_2/N_1 \to \overline{N}_2/\overline{N}_1$) works in the case of $\overline{G}$.

This definition is slightly stronger than the natural version of the condition that N_2/N_1 has cofinality ω. Indeed, our definition guarantees that the sequence of subgroups (use the fact that each U_i is an H-module) $HU_0 < HU_1 < HU_2 < ...$

is proper and the corresponding union coincides with G (i.e. G has countable cofinality). To see the former condition we easily reduce the problem to the situation when some $u \in U_{k+1} \setminus U_k$ is the product of some $h \in H$ and $u' \in U_k$. In this case $h \in N_2$ and our definition implies $h \in U_0 \subset U_k$, a contradiction.

By Proposition 3.10 of [18] any compact Polish group has uncountable topological cofinality. In the following theorem we apply the term which we have just introduced to characterize when $\overline{G}$ has countable cofinality.

Theorem 3.2. *Let G be a countably categorical group. Then the completion $\overline{G}$ has uncountable cofinality if and only if G is residually finite and the following properties hold:*
(1) in any series $1 = G_0 < G_1 < ... < G_n = G$ of characteristic subgroups in G, with each G_i/G_{i-1} characteristically simple, the infinite quotient G_m/G_{m-1} with maximal m is isomorphic to some Boolean power P^B with finite simple non-abelian P;
(2) for any abelian cover $N_1 < N_2$ in lattice of characteristic subgroups of G if N_2 is of infinite index in G, then the $\overline{G}/\overline{N}_2$-module $\overline{N}_2/\overline{N}_1$ does not have countable $\overline{G}/\overline{N}_2$-cofinality.

We start with some preliminary lemma.

Lemma 3.3. *Let P be a finite simple non-abelian group, V_1 (V_0 resp.) be the unique countable atomless Boolean ring with (without) identity and $G = P^{V_1}$ ($G = P^{V_0}$ resp.). Then the profinite completion $\overline{G}$ is isomorphic to the direct power P^{ω}.*

Proof. We apply Proposition 5.1 of [1], which describes normal subgroups of Boolean powers. According to this result the normal subgroups of G are just those of the form P^S, for S an ideal of $B = V_1$ ($= V_0$ resp.). It is easy to see that this subgroup is the kernel of the homomorphism $P^B \to P^{B/S}$ mapping each $(g_1)_{A_1} \cdots (g_k)_{A_k}$ (with $A_1, ..., A_k$ disjoint) to $(g_1)_{A_1+S} \cdots (g_k)_{A_k+S}$. When $P^{B/S}$ is finite, it is naturally isomorphic to the group P^m where m is the number of atoms of B/S. Now it is easy to see that the corresponding inverse system consists of projections between groups of the form P^m. The corresponding limit is the direct power P^{ω}. $\square$

We will also use the following theorem of S.Koppelberg and J.Tits [11]:

> Let F be a non-trivial finite group. Then the power F^ω has uncountable cofinality if and only if F is perfect.

Proof of Theorem 3.2. Necessity follows from Theorem B of [2] and Proposition 3.1. Indeed assuming the contrary we are left to the case when G is residually finite (by Proposition 3.1(1)) and condition (2) is satisfied (see the discussion above concerning countable G/N_2-cofinality). Theorem B of [2] guarantees that G_m/G_{m-1} is not an infinite non-abelian characterically simple p-group. Thus, when (1) does not hold, $\overline{G}$ has a subgroup of finite index with an infinite abelian quotient. Now Proposition 3.1(2) implies that this quotient and $\overline{G}$ have countable cofinality.

To prove sufficiency consider $1 = G_0 < G_1 < ... < G_n = G$, any series of characteristic subgroups in G, with each G_i/G_{i-1} characteristically simple. Find $m \leq n$ such that $|G : G_m|$ is finite (say k) and $|G_m : G_{m-1}|$ is infinite. By our assumptions the infinite quotient G_m/G_{m-1} is isomorphic to some Boolean power P^B with finite simple non-abelian P and $B = V_0$ or $= V_1$.

Consider all finite quotients of G. If L is a finite quotient of G with $|G : G_m| < |L|$, then the index of the image of G_m in L is also bounded by k. Thus the closure $\overline{G}_m$ in $\overline{G}$ is of index k in $\overline{G}$ and the closure $\overline{G}_{m-1}$ is a normal subgroup such that $\overline{G}/\overline{G}_{m-1}$ is a profinite completion of G/G_{m-1} (by Lemma 2.2). By Lemma 3.3 the quotient $\overline{G}_m/\overline{G}_{m-1}$ is of the form P^ω for P as above.

We now assume that $\overline{G} = \bigcup H_i$ for some ω-chain $1 < H_1 < ... < H_i < ...$ of subgroups of $\overline{G}$. Choose the minimal $l \leq m$ with the property that for some $i \in \omega$, $H_i\overline{G}_l = \overline{G}$. It is clear that we may assume that $l > 0$. Consider the sequence $(H_i\overline{G}_{l-1})/\overline{G}_{l-1} \cap \overline{G}_l/\overline{G}_{l-1}$, $i \in \omega$. We may assume this sequence is strictly increasing and $H_1\overline{G}_l = \overline{G}$.

If $\overline{G}_l/\overline{G}_{l-1}$ is abelian, then $l < m$, i.e. $\overline{G}/\overline{G}_l$ is infinite. Representing $\overline{G}/\overline{G}_l$ as $H_1\overline{G}_l/\overline{G}_l$ we easily see that each $(H_i\overline{G}_{l-1})/\overline{G}_{l-1} \cap \overline{G}_l/\overline{G}_{l-1}$ is a $\overline{G}/\overline{G}_l$-module. Taking $U_i = H_i\overline{G}_{l-1} \cap \overline{G}_l$ we obtain a contradiction with property (2) of the formulation.

If $\overline{G}_l/\overline{G}_{l-1}$ is not abelian then we see as above that $\overline{G}_l/\overline{G}_{l-1}$ is isomorphic to some product of the form P^ω, where P is finite, simple and non-abelian. On the other hand the sequence $(H_i\overline{G}_{l-1})/\overline{G}_{l-1} \cap \overline{G}_l/\overline{G}_{l-1}$, $i \in \omega$, covers $\overline{G}_l/\overline{G}_{l-1}$, i.e.

$\overline{G}_l/\overline{G}_{l-1}$ has countable cofinality. This contradicts a theorem of Koppelberg and Tits (see [11]). □

The methods applied above can also work for some other questions naturally arising in our approach. Assume that G is an ω-categorical, residually finite group. As we already know, for any normal subgroups $N_1 < N_2 < G$ the quotient $\overline{N}_2/\overline{N}_1$ of their closures in the completion $\overline{G}$ is the profinite completion of N_2/N_1. As G is residually finite, the lattice of characteristic subgroups of G does not contain any covers $N_1 < N_2$ with N_2/N_1 non-abelian and characteristically simple p-group (Theorem B of [2]). What are possibilities for $\overline{N}_2/\overline{N}_1$?

Corollary 3.4. *Assume that G is an ω-categorical, residually finite group. If for every cover $N_1 < N_2$ of the lattice of characteristic subgroup of G, the completion of N_2/N_1 can be presented as an inverse limit which is small in the sense of Newelski, then $\overline{G}$ is soluble.*

Let us recall necessary preliminaries from [17]. Let Γ be a profinite group and $G_0 \leftarrow G_1 \leftarrow ...$ be the corresponding inverse system. Any automorphism of Γ fixing all G_i (as sets) is called a *profinite automorphism* of Γ. By $Aut^*(\Gamma)$ we denote the group of all profinite automorphisms of Γ. L.Newelski defines that Γ is *small* if the number of all $Aut^*(\Gamma)$-orbits on Γ is at most countable. The main conjecture in these respects states that small profinite groups have open abelian subgroups.

Proof of Corollary 3.4 To prove the corollary it is enough to prove that Boolean products P^B with P finite, non-abelian and simple, cannot be realized by quotients of covers of the lattice (then by ω-categoricity we see that the lattice is finite and by Theorem A of [2] it has only abelian covers, thus G is soluble).

Assume that $N_1 < N_2$ is a cover such that $N_2/N_1 \cong P^B$, where P is as above. Then by Theorem A of [2] B is an atomless Boolean ring, i.e. V_0 or V_1 in our notation above. By Lemma 3.3 the profinite completion of N_2/N_1 is isomorphic to the product P^ω. By Remark 4.2 of [17], the group P^B is not small. □

4. Categoricity of the completion

Let G be an ω-categorical group. We will assume here that G is soluble. Then by the main theorem of [3] G is nilpotent-by-finite or interprets the countable atomless

Boolean algebra. In the former case $\overline{G}$ is nilpotent-by-finite too. In this section we show that the latter case is impossible if G is residually finite and $Th(\overline{G})$ is ω-categorical.

We start our investigations with an observation that tricks of the previous sections are still helpful for the question when $Th(\overline{G})$ is ω-categorical (in fact we will use them in the main theorem of the section). Consider ω-categorical residually finite G acting on $X = \{\xi_1, \xi_2, ..., \xi_n, ...\}$ under the action defined as in Section 2. We present X as an increasing sequence of G-invariant subsets $X_n = G\xi_1 \cup ... \cup G\xi_n$, $n \in \omega$. Then $\overline{G}$ is the inverse limit of the permutation groups (G_i, X_i) induced on all X_i by G and with appropriate projections. Lemma 2.2 describes the case when $N_1 < N_2 < G$ are normal subgroups of G. Then by Lemma 2.2 the inverse system $\{... \leftarrow (G_i \cap N_2)/(G_i \cap N_1) \leftarrow ...\}$ induced by $\{... \leftarrow (G_i, X_i) \leftarrow ...\}$ has the limit isomorphic to the quotient $\overline{N}_2/\overline{N}_1$ of the corresponding closures in $\overline{G}$.

Proposition 4.1. *Le G be an ω-categorical residually finite group and $N_1 < N_2 < G$ be a cover in the lattice of characteristic subgroups. If $Th(\overline{G})$ is ω-categorical and the closures $\overline{N}_1$ and $\overline{N}_2$ are definable in $\overline{G}$ then $\overline{N}_2/\overline{N}_1$ is abelian.*

Proof. If the cover $N_1 < N_2$ is not abelian, then by Theorems A and B of [2], $N_2/N_1 \equiv P^B$, where P is finite, simple and non-abelian, and B is an atomless Boolean ring, i.e. V_0 or V_1 in our notation above.

By Lemma 2.2 the quotient $\overline{N}_2/\overline{N}_1$ is the profinite completion of N_2/N_1. On the other hand by Lemma 3.3 the profinite completion of N_2/N_1 is isomorphic to the product P^ω. By the assumptions of the proposition, $\overline{N}_2/\overline{N}_1$ is definable in $\overline{G}$. Since P^ω cannot be ω-categorical [19], we see that $\overline{G}$ is not ω-categorical. □

Note that this proposition somehow suggests that in the ω-categorical residually finite case the group $\overline{G}$ must be very similar to soluble groups: if there is a series of characteristic closed subgroups as in the Apps' theorem, then $\overline{G}$ is soluble. The main theorem of the section is devoted to this case.

Theorem 4.2. *Let G be an ω-categorical, residually finite, soluble group such that the profinite completion $\overline{G}$ has ω-categorical elementary theory. Then both G and $\overline{G}$ are nilpotent-by-finite.*

The proof of this theorem uses ideas of Proposition 4.1 and the paper [3]. We will assume G is a soluble, residually finite, ω-categorical group which is not nilpotent-by-finite. Then by Lemma 2.3 $\overline{G}$ is soluble and is not nilpotent-by-finite. In order to obtain a contradiction assume that $\overline{G}$ is ω-categorical. We start with the following lemma.

Lemma 4.3. *Let G be an ω-categorical, soluble, residually finite group which is not nilpotent-by-finite. If the elementary theory of the profinite completion $\overline{G}$ is ω-categorical, then there are closed characteristic subgroups $S_1 < S_2 < \overline{G}$ and $T_1 < T_2 < \overline{G}$ such that S_2/S_1 is abelian over its centre, $[S_2/S_1, S_2/S_1]$ is finite, T_2/T_1 is an S_2/S_1-module under the conjugacy action and the corresponding semidirect product of T_2/T_1 and S_2/S_1 is not nilpotent-by-finite.*

Proof. In fact Archer and Macpherson have proved this lemma in [3] without the condition of closedness of the corresponding subgroups (they did not have any topological assumptions). We analyse $\overline{G}$ by the method of [3]. Find the minimal number r such that the quotient $\overline{G}/\overline{G}^{(r)}$ is not nilpotent-by-finite, but $\overline{G}/\overline{G}^{(r-1)}$ so is. Let $A = \overline{G}^{(r-1)}$ and let F be the Fitting subgroup of $\overline{G}/A$. By Lemma 2.4 A is the closure of $G^{(r-1)}$ and $\overline{G}^{(r)}$ is the closure of $G^{(r)}$. As FA/A is a characteristic subgroup of $\overline{G}/A$ of finite index, we also have that FA is a characteristic subgroup of $\overline{G}$ of finite index. Note that FA is the closure in $\overline{G}$ of the preimage in G of the Fitting subgroup (say F_0) of $G/G^{(r-1)}$. Indeed, by Lemma 2.2 the quotient $\overline{G}/A$ is the profinite completion of $G/G^{(r-1)}$. Thus $G/G^{(r-1)}$ is nilpotent-by-finite and $F_0G^{(r-1)}$ has the closure of finite index in $\overline{G}$. Since both $G/(F_0G^{(r-1)})$ and $\overline{G}/(FA)$ do not have normal nilpotent subgroups, the group FA is the closure of $F_0G^{(r-1)}$ in $\overline{G}$ (remember that by Lemma 2.3 the closure of a nilpotent characteristic subgroup is also nilpotent).

Let $F = Z_c > ... > Z_0 = 1$ be the lower central series of F. Then by Lemma 2.4 each Z_tA is a closed characteristic subgroup of $\overline{G}$. Find the maximal t such that $(Z_tA)/\overline{G}^{(r)}$ has a nilpotent subgroup of finite index. By our assumptions $c \neq t$. Let $S_1 \geq A$ be the preimage in $\overline{G}$ of the Fitting subgroup of $(Z_tA)/\overline{G}^{(r)}$. Thus S_1 is a characteristic subgroup of $\overline{G}$. Applying arguments as in the previous paragraph we see that S_1 is closed.

Since $S_1/\overline{G}^{(r)}$ is nilpotent there is a sequence of S_1-invariant closed subgroups $\overline{G}^{(r)} = A_0 < ... < A_n = A$, characteristic in $\overline{G}$, such that S_1/A_0 centralizes each A_{i+1}/A_i. Here we assume that each A_i is the intersection of A with the corresponding member of the lower central series of $S_1/G^{(r)}$ (thus we can apply Lemma 2.4). By a straightforward argument (see the induction step in the proof of Lemma 2.1 of [3]) there is i such that the semidirect product of A_{i+1}/A_i and $(Z_{t+1}A)/S_1)$ (under the conjugacy action) is not nilpotent-by-finite. Let $T_1 = A_i$, $T_2 = A_{i+1}$ and $S_2 = Z_{t+1}A$. As we already mentioned these subgroups are closed and characteristic in $\overline{G}$.

Note that the centralizer of $(Z_tA)/S_1$ in $(Z_{t+1}A)/S_1$ defines a characteristic closed subgroup $U < \overline{G}$ of finite index in $Z_{t+1}A$. Thus we may assume that S_2/S_1 centralizes $(Z_tA)/S_1$ (replacing $Z_{t+1}A$ by U if necessary). $\square$

In the situation of the lemma consider the action of S_2/S_1 on T_2/T_1 in more detail. Since the semidirect product of T_2/T_1 and S_2/S_1 is not nilpotent-by-finite, the pointwise stabilizer of T_2/T_1 (fixator) of the conjugacy action of S_2/S_1 is of infinite index in the latter. Since T_1 and T_2 are characteristic in $\overline{G}$, the conjugacy-fixator of T_2/T_1 is also characteristic in $\overline{G}$. Since it is also closed we can enlarge S_1 so that S_2/S_1 acts on T_2/T_1 faithfully.

Decomposing the profinite group S_2/S_1 into the direct sum of (finitely many) its Sylow subgroups we find a prime number r such that the preimage in $\overline{G}$ of the r-Sylow subgroup (which is obviously characteristic and closed) satisfies all the assumptions of the lemma above for S_2.

On the other hand decomposing the profinite group T_2/T_1 into the direct sum of (finitely many) its Sylow subgroups we find a prime number p and the corresponding p-Sylow subgroup P such that for the natural action of S_2/S_1 on P the corresponding semidirect product is not nilpotent-by-finite. Extending T_1 by the preimage in $\overline{G}$ of the complement of P (which is obviously characteristic and closed) we still have that all the assumptions of the lemma above are satisfied.

Note that when $p = r$, the corresponding semidirect product is locally nilpotent, and thus by ω-categoricity and [21], is nilpotent.

We now summarize the situation as follows.

Lemma 4.4. *Let G be an ω-categorical, soluble, residually finite group which is not nilpotent-by-finite. Let the completion $\overline{G}$ be ω-categorical too. Then there are closed characteristic subgroups $S_1 < S_2 < \overline{G}$ and $T_1 < T_2 < \overline{G}$ such that for two prime numbers $r \neq p$, S_2/S_1 is an r-group which is abelian over its center, T_2/T_1 is an abelian p-group which is a faithful S_2/S_1-module under the conjugacy action and the corresponding semidirect product of T_2/T_1 and S_2/S_1 is not nilpotent-by-finite. Moreover $[S_2/S_1, S_2/S_1]$ is finite.*

The following lemma is the main preliminary step to Theorem 4.2.

Lemma 4.5. *Under the assumptions of Lemma 4.4 there is a finite tuple $\bar{w} \in \overline{G}$ and there are closed $\bar{w}$-definable subgroups $S_1 < S_2 < \overline{G}$ and $T_1 < T_2 < \overline{G}$ such that for two prime numbers $r \neq p$, S_2/S_1 is an infinite abelian r-group, T_2/T_1 is an elementary abelian p-group which is a faithful S_2/S_1-module under the conjugacy action.*

Proof. In the situation provided by Lemma 4.4 let p^n be the exponent of $P = T_2/T_1$ and for each $i = 0, ..., n-1$ let $P_i = p^i P$ and K_i be the centralizer of P_i/P_{i+1} in S_2/S_1. Since all $P_i T_1$ are characteristic and closed in $\overline{G}$, each $K_i S_1$ is characteristic and closed too. If all $K_i S_1$ are of finite index in S_2, then $(\bigcap K_i)$ is an r-group centralizing the p-group P (Theorem 5.3.2 of [9]) contradicting the assumption that the action is faithful. Replacing P by an appropriate P_i/P_{i+1} and S_2/S_1 by the corresponding $S_2/(K_i S_1)$ we arrange that T_2/T_1 is a vector space over $GF(p)$. Denote $V = T_2/T_1$.

We now follow the proof of Lemma 2.7 of [3]. Our strategy is to show that at every step of that proof we build closed subgroups of $\overline{G}$.

Assume that S_2/S_1 is not abelian. Let K be the largest finite characteristic subgroup of S_2/S_1. Then according to the construction from the proof of Lemma 4.3 we have $[S_2/S_1, S_2/S_1] \leq K$. If the centre of S_2/S_1 is infinite the proof is finished by replacing S_2/S_1 by its centre. Thus passing to the centralizer of K if necessary, we may assume that K is the centre of S_2/S_1.

By Maschke's theorem we may write V as a direct sum of $GF(p)K$-irreducible submodules. Since the number of isomorphism types of such irreducibles is finite, by collecting them in this sum into isomorphism classes we may suppose that $V =$

$W_1 \oplus ... \oplus W_t$, where each W_i is a direct sum of isomorphic $GF(p)K$-irreducibles, and for $i \neq j$ irreducibles appearing in W_i and W_j are not isomorphic. Note that each W_iT_1 can be easily presented as an intersection of clopen subgroups of $\overline{G}$ (appearing from appropriate finite quotients of $\overline{G}$). By a similar argument any subgroup of the form $(W_{i_1} \oplus ... \oplus W_{i_l})T_1$ is closed.

We may suppose that V is a direct sum of isomorphic faithful irreducible $GF(p)K$-modules. To see this we apply the following reductions of Lemma 2.7 of [3]. For $i = 1, ..., t$ let $C_i = C_{S_2/S_1}(W_i)$. It is easy to see that some group $S_2/(C_iS_1)$ is infinite (using the fact that T_2/T_1 is a faithful S_2/S_1-module) and each C_iS_1 is closed in $\overline{G}$. It is also clear that the group $(\sum\{W_i : S_2/(C_iS_1)$ is infinite $\})T_1$ is characteristic and closed in $\overline{G}$. We thus can identify it with T_2. If some $S_2/(C_iS_1)$ has infinite centre we replace V by W_i and S_2/S_1 by the centre of $S_2/(C_iS_1)$. We thus may suppose that each $S_2/(C_iS_1)$ has finite centre. Now replace V by an appropriate W_i and S_2/S_1 by the corresponding $S_2/(C_iS_1)$.

Now by Schur's lemma, each non-trivial element of K acts fixed-point-freely on V. The second claim of the proof of Lemma 2.7 from [3] states that there exists a non-trivial $w \in V$ such that the centralizer $C_{S_2/S_1}(w)$ is infinite. Since $K \setminus \{1\}$ acts fixed-point-freely on on $V \setminus \{0\}$, the centralizer $C_{S_2/S_1}(w)$ is disjoint from $K \setminus \{1\}$. Since S_2/S_1 is nilpotent of class two, $C_{S_2/S_1}(w)$ is abelian. Since $C_{S_2/S_1}(w)S_1$ is closed in $\overline{G}$ we can replace (adding a constant) S_2 by $C_{S_2/S_1}(w)S_1$. $\square$

Proof of Theorem 4.2. Let G be an ω-categorical, soluble, residually finite group which is not nilpotent-by-finite. Let the completion $\overline{G}$ be ω-categorical too. As we have already known by adding finitely many constants we can arrange that there are closed characteristic subgroups $S_1 < S_2 < \overline{G}$ and $T_1 < T_2 < \overline{G}$ such that for two prime numbers $r \neq p$, S_2/S_1 is an abelian r-group, $V = T_2/T_1$ is an elementary abelian p-group which is a vector space over $GF(p)$ and a faithfull $GF(p)[S_2/S_1]$-module under the conjugacy action.

Furthermore by picking some additional parameters we may assume that

there is $v \in V$ such that $V = \langle v^{S_2/S_1} \rangle$.

Indeed, if for every $v \in V$ the centralizer $C_{S_2/S_1}(v)$ is of finite index in the abelian r-group S_2/S_1, then by Proposition 3.4 of [14] [5] (applied to the semidirect product as above) the group S_2/S_1 cannot act faithfully on V. Thus choosing $v \in V$ with infinite vector space $\langle v^{S_2/S_1}\rangle$ we see that this space is definable over v (by ω-categoricity) and is of the form T_3/T_1 where $T_3 < T_2$ is closed (using that $\overline{G}$ is compact we see that the S_2/S_1-orbit of v is closed). To have a faithful action on V we now replace S_2/S_1 by its quotient over the centralizer of v.

By ω-categoricity there is $n \geq 1$ such that each element of V can be written as a sum of at most n S_2/S_1-translates of v. We introduce ring operations $\oplus$ and $\otimes$ on V as follows. Let $\oplus$ be the group operation on V. To define $\otimes$ let $v = \sum_{1\leq i\leq r} v^{h_i}$ and $v' = \sum_{1\leq i\leq s} v^{h'_i}$ be from V, where $h_i, h'_i \in S_2/S_1$. Then define $v \otimes v'$ to be $\sum_{1\leq i\leq r}\sum_{1\leq j\leq s} v^{h_i h'_j}$. It is proved in [3] (Corollary 3.2) that V becomes a definable ring without non-zero nilpotent elements.

Note that if ϕ is a homomorphism from $\overline{G}$ to a finite group G_1, then the action in G_1 induced by ϕ and the action of S_2/S_1 on T_2/T_1, defines as above a finite ring without non-zero nilpotent elements (the argument of Corollary 3.2 from [3] works in this case too). Moreover since $\overline{G}$ is profinite the action of S_2/S_1 on T_2/T_1 is the inverse limit of the system of all such actions of finite groups. Thus the ring V is the inverse limit of the system of corresponding finite rings.

We now analyse the structure of $(V, \oplus, \otimes)$ using the Macintyre-Rosenstein description of ω-categorical rings from [13]. According this description V is a direct product of finitely many 0-definable rings R_i and each R_i is isomorphic to a filtered Boolean power of a finite field F_i with respect to an appropriate Boolean space X_i and a map τ_i to a lattice of subfields of F_i (the corresponding augmented Boolean algebras B_i are interpretable in V and thus ω-categorical).

Let I be an ideal of B_i such that B_i/I is finite. Then it is easy to see that the filtered Boolean power of F_i with respect to B_i/I and the corresponding "quotient" of τ_i is a product of finite direct powers of subfields from $Rng(\tau_i)$. As in the proof of Lemma 3.3 we see that this product is a homomorphic image of $C(X_i, F_i, \tau_i)$. By the ring theory version of Proposition 5.1 from [1] (see Introduction) any homomoprphic

[5] if G is an ω-categorical group with a characteristic subgroup V which is a vector space over $GF(p)$ with G/V a nilpotent group with no elements of order p, and with V a sum of finite-dimensional $GF(p)[G/V]$-modules, then the G/V-centralizer of V is of finite index in G/V

image of $C(X_i, F_i, \tau_i)$ is of this form. Thus the inverse limit of all finite quotients of $C(X_i, F_i, \tau_i)$ is the direct product of F_i^{ω} and finitely many direct powers $\tau_i(C)^{\omega}$, $C \in Rng(\tau_i)$.

From this description we see that $(V, \oplus, \otimes)$ is the direct product of finitely many direct powers R^{ω}, where R is a subfield of some F_i as above. This is a contradiction. To see this precisely note that R^{ω} cannot be ω-categorical. Let $f_1, f_2, ..., f_i, ...$ be a sequence of elements of R^{ω} with finite support such that for any pair of distinct i, j we have $|supp(f_i)| \neq |supp(f_j)|$. Then all ideals $f_i R^{\omega}$ are of pairwise distinct size. We see that for all $i \neq j$, the elements f_i and f_j have distinct types, which contradicts ω-categoricity. $\square$

Let G be an ω-categorical group having a normal abelian subgroup A of finite index. Consider A as a G/A-module. Then A is decomposed into a direct sum of finitely many modules A_i such that each A_i is a direct sum of ω copies of a finite indecomposable G/A-module (see [6] and Sections 11,12 and Apendix of [7]). It is clear that the profinite completion of the module A is an elementary extension of A. This implies that

> when G is an ω-categorical abelian-by-finite group, G is residually finite and is an elementary substructure of the completion $\overline{G}$.

In particular under the BCM-conjecture [6] we have that a stable ω-categorical group is residually finite and is an elementary substructure of its profinite completion. Krzysztof Krupiński has suggested that this statement can be proved without the BCM-conjecture (and then this can be considered as a small confirmation of the BCM-conjecture). We think that an appropriate development of the methods of [14] allows a stronger statement: an ω-categorical group without the strict order property is residually finite and is an elementary substructure of its profinite completion. Since the technique of [14] is quite involved we postpone this for a separate reseach.

[6] an ω-categorical stable group is abelian-by-finite [7]

5. Measuring the set of commuting pairs

Let G be an ω-categorical group, H be an inert residually finite subgroup and $\overline{G}$ be the completion of G with respect to $X = \{gK : K$ is commensurable with H and $g \in G\}$. Let μ be the Haar measure on $\overline{G}$. In the case when $\overline{G}$ is profinite and μ is normalized it follows from the main result of [12] that if there is a real number $\varepsilon > 0$ such that $(\mu \times \mu)(\{(x,y) \in \overline{G} \times \overline{G} : xy = yx\}) = \varepsilon$, then $\overline{G}$ is abelian-by-finite. In this section we consider some related conditions which work in the case of $\overline{G}$ without the assumption that $\overline{G}$ is profinite. We also describe some approximating functions which naturally arise in these respects.

Let $Y \subset X$ consist of finitely many G-orbits. By G_Y ($\overline{G}_Y$ resp.) we denote the pointwise stabilizer of Y in G (in $\overline{G}$). We denote $G(Y) := G/G_Y$ and $\overline{G}(Y) := \overline{G}/\overline{G}_Y$. These groups acts faithfully on Y and $\overline{G}(Y)$ is the completion of $G(Y)$ under the topology of this action. As in Proposition 2.3 we see that every finitely generated subgroup of $\overline{G}(Y)$ is a homomorphic image of a finite subgroup of $G(Y)$.

Let $\rho_{com}(i)$ be the function which associates to each natural number i the maximal number n such that if L is a subgroup of some $G(Y)$ where $Y \subset X$ consists of finitely many G-orbits, and $|L| = i$, then $n \leq |\{(x,y) \in L \times L : xy = yx\}|$. If no $G(Y)$ has subgroups of size i, then we put $\rho_{com}(i) = \infty$.

In the case when G is residually finite and X consists of cosets of subgroups of finite index, the function $\rho_{com}(i)$ is closely connected with the value $(\mu\times\mu)(\{(x,y) \in \overline{G} \times \overline{G} : xy = yx\})$. On the other hand it is clear that the square $\mu \times \mu$ is equal to the normalized Haar measure on $\overline{G} \times \overline{G}$. We will denote it by the same letter μ. By a straightforward generalization of (the first proof of) Theorem 1(iii) from [12] we have the following statement.

Lemma 5.1. *Let C be a compact group. If for some positive constant $\varepsilon \in \mathbf{R}$, $\mu_{C\times C}(\{(x,y) \in C \times C : xy = yx\}) > \varepsilon$, then C is finite-by-abelian-by-finite.* [7]

Proof. The proof of Lemma 5 from [12] without the last two sentences, works for any compact group [8] and shows that if the FC-centre of C (i.e. the set of elements of C having finitely many conjugates) is of finite index in C then C is

[7] in [12] this statement is formulated for profinite groups but with a slightly stronger conclusion: C is abelian-by-finite

[8] in [12] it concerns profinite groups

finite-by-abelian-by-finite. Now the first proof of Theorem 1(iii) [12] works in our case. □

Below we also use another observation from [12] (see Theorem 1(ii) and its proofs). This is the fact that in the profinite case the condition $\mu(\{(x,y) \in C \times C : xy = yx\}) = \varepsilon$ is equivalent to the condition that there are natural numbers n_1 and n_2 such that for any finite quotient L of C there is a subgroup $N < L$ with $|L : N| \leq n_1$ and $|[N, N]| \leq n_2$.

Lemma 5.2. *Let G be a residually finite group. Assume that there is a positive real number $\varepsilon < 1$ such that for any finite quotient L of G we have $\rho_{com}(|L|) \geq \varepsilon |L|^2$. Then both G and the profinite completion $\overline{G}$ are abelian-by-finite and there is an open normal subgroup $N < \overline{G}$ of nilpotency class two such that $|[N, N]|$ and $|\overline{G}/N|$ are both ε bounded.*

In particular if G is finite-by-abelian-by-finite (for example, if $Th(G)$ is ω-categorical and supersimple [8]*), then both G and $\overline{G}$ are abelian-by-finite.*

Proof. Assume that for any finite quotient L of G, $|\{(x, y) \in L \times L : xy = yx\}| \geq \varepsilon |L|^2$. Since the set of commuting pairs is closed in $\overline{G} \times \overline{G}$ we easily see that the μ-measure of this set is at least ε. By Theorem 1 of [12] we obtain that $\overline{G}$ is abelian-by-finite, and there is an open normal subgroup $N < \overline{G}$ of nilpotency class two such that $|[N, N]|$ and $|\overline{G}/N|$ are both ε bounded. Since G embeds into $\overline{G}$, G is abelian-by-finite too.

To see the second statement of the lemma note that the assumptions of this statement imply that there is a real number $\varepsilon < 1$ such that for any finite quotient L of G there are normal subgroups $K < N < L$ such that N/K is abelian and $|K| \cdot |L : N|^2 \leq \varepsilon^{-1}$. Applying the inequality from [16] (see Section 1)

$$|\{(x, y) \in L \times L : xy = yx\}| \geq |L|^2/(|K| \cdot |L : N|^2)$$

we see that $|\{(x, y) \in L \times L : xy = yx\}| \geq \varepsilon |L|^2$. Thus we can apply the basic statement of the lemma. □

We now consider the case when G is locally finite and $\overline{G}$ is locally compact. The following proposition is the main result of the section.

Proposition 5.3. *Let G be a locally finite group having finite exponent and infinite inert residually finite subgroups (for example G is ω-categorical). Let X be defined as above for some infinite inert residually finite $H \leq G$. The following conditions are equivalent.*
(1) There is a positive real number $\varepsilon < 1$ such that for any finite subgroup L of $G(Y)$ with $Y \subset X$ consisting of finitely many G-orbits, we have $\rho_{com}(|L|) \geq \varepsilon |L|^2$;
(2) There is a positive real number $\varepsilon < 1$ such that for any infinite compact subgroup $C < \overline{G}$ we have $\mu_C(\{(x,y) \in C \times C : xy = yx\}) \geq \varepsilon$, where μ_C is the normalized Haar measure on C;
(3) Both G and $\overline{G}$ are finite-by-abelian-by-finite.

Each of the conditions (1), (2) and (3) implies that every infinite inert residually finite subgroup $K < G$ (and the completion $\overline{K}$) is abelian-by-finite.

The group G is abelian-by-finite if and only if there is a number l such that every infinite inert residually finite subgroup $K < G$ (and the completion $\overline{K}$) is abelian-by-$\leq l$.

Proof. (1) $\Rightarrow$ (3). We will use the following theorem of P.Neumann. For each positive number ε there are natural numbers n_1 and n_2 such that if a finite group F has at least $\varepsilon |F|^2$ commuting pairs, then there is a normal subgroup $N < F$ with $|F : N| \leq n_1$ and $|[N,N]| \leq n_2$ [16]. Moreover in this case $n_1^2 n_2 \leq \varepsilon^{-1}$.

Let $Y \subset X$ consist of finitely many G-orbits and $L < G(Y)$ be finite. Then by Neumann's theorem there is a normal subgroup $N < L$ such that $|L : N| \leq n_1$ and $|[N,N]| \leq n_2$ where n_1, n_2 are determined by ε as above.

Now note that this statement holds for any finite subgroup of G. This follows from the fact that if F is a finite subgroup of G, then F can be realized as a subgroup of $G(Y)$ for an appropriate Y as above. Indeed since H is inert, F normalizes a subgroup $H_1 < H$ of finite index in H such that $F \cap H_1 = \{1\}$. Then for $Y = \{gH_1 : g \in G\}$ (which is a G-orbit) there is an embedding of F into $G(Y)$. Thus we can find a normal subgroup $N < F$ such that $|F : N| \leq n_1$ and $|[N,N]| \leq n_2$.

We present G as the union of an ω-chain of finite groups $L_0 < L_1 < ... < L_i < ....$ For each i we find a normal $N_i < L_i$ with $|L_i : N_i| \leq n_1$ and $|[N_i, N_i]| \leq n_2$. Choosing a subsequence if necessary we can arrange that for any pair $g_1, g_2 \in G$

there is a natural number j such that we have $(\forall i \geq j) L_i \models (g_1 N_i = g_2 N_i)$ or $(\forall i \geq j) L_i \models (g_1 N_i \neq g_2 N_i)$. As a result we see that $N_\omega := \{g \in G : g \in N_i$ for infinitely many $i\}$ is a subgroup of index $\leq n_1$ in G. Since each $|[N_i, N_i]| \leq n_2$ we see that $[N_\omega, N_\omega] \leq n_2$. This obviously means that $\overline{N}_\omega$ is finite-by-abelian. Since $\overline{N}_\omega$ is of index $\leq n_1$ in $\overline{G}$, this proves (3).

Let us prove (3) $\Rightarrow$ (2). Find n_1 and n_2 such that there is a normal subgroup $N < G$ of index $\leq n_1$ such that $|[N, N]| \leq n_2$. Let C be a compact subgroup of $\overline{G}$. Then C and all its finite quotients are ($\leq n_2$)-by-abelian-by-($\leq n_1$). This implies that there is a positive real number $\varepsilon < 1$ such that for any finite quotient L of C we have $|\{(x, y) \in L \times L : xy = yx\}| \geq \varepsilon |L|^2$ (by an easy inequality from Section 1 of [16] we may put $\varepsilon = (n_1^2 n_2)^{-1}$). This implies (2).

(2) $\rightarrow$ (1). Note that if H_0 is a residually finite inert subgroup of G which is commensurable with H, then the closure of H_0 in $\overline{G}$ is profinite. Condition (2) implies that $\overline{H}_0$ satisfies the assumption of Theorem 1 of [12]. Thus $\overline{H}_0$ has an open normal subgroup $N < \overline{H}_0$ of nilpotency class two such that $|[N, N]|$ and $|\overline{H}_0 / N|$ are both ε bounded.

On the other hand if F is a finite subgroup of G, then F can be realized as a subgroup of an inert residually finite H_0 which is commensurable with H. Indeed since H is inert, F normalizes a subgroup $H_1 < H$ of finite index in H. Then $\langle H_1, F \rangle$ works as H_0. The fact that $\langle H_1, F \rangle$ is inert and commensurable with H is obvious. To see residual finiteness note that if H_2 is a normal subgroup of H_1 of finite index, then it is of finite index in $\langle H_1, F \rangle$ and thus contains a normal subgroup of $\langle H_1, F \rangle$ of finite index. Since H_1 is residually finite, $\langle H_1, F \rangle$ is residually finite too.

Thus as in the proof of Lemma 5.2 for every finite subgroup $F < G$ we can find a normal subgroup $N < F$ such that both $|[N, N]|$ and $|F : N|$ are ε bounded. By an argument from [16] (or the proof of Lemma 5.2) for any finite subgroup L of $G(Y)$ with $Y \subset X$ consisting of finitely many G-orbits, we have (correcting ε if necessary) $\rho_{com}(|L|) \geq \varepsilon |L|^2$.

The statement of the proposition that all inert residually finite subgroups are abelian-by-finite, follows from Lemma 5.2.

To see the last statement note that the condition that there is a number l such that every infinite inert residually finite subgroup $K < G$ (and the completion $\bar{K}$) is abelian-by-$\leq l$ implies that every finite subgroup of G is abelian-by-$\leq l$ (see the proof of (2) $\Rightarrow$ (1)). Now applying the argument of (1) $\Rightarrow$ (3) we see that G is abelian-by-$\leq l$. $\square$

The function ρ_{com} is connected with some other functions naturally arising in these respects. For example consider the following asymptotic condition. Let $\rho_r(i)$ be the function (defined by G) which associates to each natural number i the minimal number m such that whenever L is a quotient of G with $|L| = i$, then H/H_L has rank at most m for all subgroups H of L. Here H_L is the maximal normal subgroup of L contained in H (*the core of H in L*). It is worth mentioning that Proposition 4.3 of [10] states that there is a function $\beta : \omega \to \omega$ such that if L as above is a finite group with H/H_L of rank at most r for all $H < L$, then L is abelian-by-(group of rank $< \beta(r)$). Assuming that G is locally finite of exponent $m \in \omega$ we can apply Zelmanov's solution of the restricted Burnside problem to derive that in fact the function β can be chosen so that L is abelian-by-(group of order $< \beta(r)$).

Another interesting function can be defined using the approach of Baudisch (in [4]) to the BCM-conjecture. Assume that G is additionally a two-step-nilpotent group and $U := G/[G,G]$. In the situation of [4] it is also assumed that U and $W := [G,G]$ are vector spaces over $GF(p)$ (this case is basic for the BCM-conjecture). Then the commutator $[x,y]$ becomes an alternating bilinear map $U \times U \to W$ over $GF(p)$. If L is a quotient of G of size i, then there is a vector space homomorphism f_L from the exterior square $\Lambda^2(L/(L \cap W))$ to $W \cap L$ [9] which makes commutative the diagram consisting of the maps $[,] : L/(L \cap W) \times L/(L \cap W) \to L \cap W$ and $\wedge : L/(L \cap W) \times L/(L \cap W) \to \Lambda^2(L/(L \cap W))$. By $\rho_\wedge(i)$ we denote the maximal k such that $k \cdot dim(L/(L \cap W)) \leq dim(Ker(f_L))$ for all L with $|L| = i$. Baudisch studied in [4] the related condition that there is a natural number k such that $dim(Ker(f_H)) \leq k \cdot dim(H/(H \cap W))$ for all finite subgroups H of G (in this case Baudisch says that G *has few relations*).

[9]if $L = G/N$, then by $L \cap W$ we denote the elements of the form wN, $w \in W$

The following computations demonstrate some connections of these functions with $\rho_{com}(i)$.

Lemma 5.4. *(1) If the function β is chosen as above, then $\beta(\rho_r(i))^{-2} \cdot i^2 \leq \rho_{com}(i)$. (2) In the situation of the function $\rho_\wedge$ we have $i^{(2\rho_\wedge(i)+1-log_p i)/2} \cdot i^2 \leq \rho_{com}(i)$.*

Proof. By an argument from [16] (p.456) for any finite group L with normal subgroups $K < N < L$ such that N/K is abelian we have the following inequality: $|\{(x,y) \in L \times L : xy = yx\}| \geq (|K| \cdot |L : N|^2)^{-1}|L|^2$. Assuming that the function β is as above, this inequality shows that: $\beta(\rho_r(i))^{-2} \cdot i^2 \leq \rho_{com}(i)$.

In the case of the function $\rho_\wedge$ we preserve the notation above. Then the argument of the previous paragraph shows that $\rho_{com}(i) \geq i^2 \cdot |W \cap L|^{-1}$. On the other hand when L is a quotient of G of size i we have $|W \cap L| = p^{dim(W \cap L)}$, $|L/(W \cap L)| = p^{dim(L/(W \cap L))}$ and $dim(W \cap L) = dim(\Lambda^2 L/(L \cap W)) - dim(Ker(f_L))$. Since $dim(\Lambda^2 L/(L \cap W)) = dim(L/(L \cap W)) \cdot (dim(L/(L \cap W)) - 1)/2$ and $dim(Ker(f_L)) \geq \rho_\wedge(i) dim(L/(L \cap W))$, we have $dim(W \cap L) \leq dim(L/(L \cap W))(dim(L/(L \cap W)) - 1 - 2\rho_\wedge(i))/2 \leq (log_p i)(log_p i - 1 - 2\rho_\wedge(i))/2$. Thus $|W \cap L|^{-1} \geq p^{(log_p i)(2\rho_\wedge(i)+1-log_p i)/2}$, which implies the required inequality. □

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